Characterizations of monadic NIP

Samuel Braunfeld
(Joint with Chris Laskowski)

University of Maryland, College Park

April 8, 2021

Overview

Definition and examples

Definition

A theory T is *monadically stable/NIP* if any expansion of T by arbitrarily many unary predicates remains stable/NIP.

- Analysis of monadically stable theories is due to Baldwin and Shelah [1].
- Refining equivalence relations and mutually algebraic theories are monadically stable
- DLO and various tree-like theories are monadically NIP.
- Essentially anything with a non-unary function is not monadically NIP, e.g. vector spaces.

CHARACTERIZATIONS

Theorem (Baldwin-Shelah [1])

The following are equivalent.

1. *T is monadically stable.*
2. *T is stable and monadically NIP.*
3. *T is stable and does not admit coding.*
4. *Models of T admit a nice decomposition into trees of countable models.*
5. *T is stable and if $B \mathop{\smile\!\!\!\!\!\!|}_D C$, then for any a, $aB \mathop{\smile\!\!\!\!\!\!|}_D C$ or $B \mathop{\smile\!\!\!\!\!\!|}_D aC$.*

Tree decompositions

Definition

A *tree decomposition* of M is a collection of countable submodels of M, indexed by a tree, such that

1. $\bigcup M_i = M$
2. If $i < j$ then $M_i \subset M_j$.
3. The children of a model M_i are independent over M_i.

- Example: An equivalence relation with κ classes of size λ.

- The children of M_i form a congruence over M_i.

(NON-)FORKING

- Recall: T is stable and if $A \mathop{\smile\!\!\!\!\!\!|}_D B$, then for any c, $cA \mathop{\smile\!\!\!\!\!\!|}_D B$ or $A \mathop{\smile\!\!\!\!\!\!|}_D Bc$.
- Equivalently, forking is trivial (i.e. if $A \not\mathop{\smile\!\!\!\!\!\!|}_C B$, then $a \not\mathop{\smile\!\!\!\!\!\!|}_C b$ for some $a \in A, b \in B$) and transitive on singletons.
- So forking defines an equivalence relation on singletons.
- Can use this equivalence relation to build the tree decomposition. (Or can use first characterization to iteratively extend by one point).

- Given a a, b, c failing this property, take Morley sequences in a and b and automorphic images of c to get coding, as in vector spaces. (c_{ij} behaves non-generically over $a_i b_j$.)

SHELAH'S THEOREM

- Soon afterward, Shelah analyzed monadic NIP [4].
- Concerned with structure theory, since non-structure was clear in his setting.

Definition

Let $A \mathop{\smile\!\!\!\!\!|}^{fs}_{M} B$ mean that $tp(A/MB)$ is finitely satisfiable in M.
Let $A \mathop{\smile\!\!\!\!\!|}^{fs}_{M \subset C} B$ mean that $tp(A/CB)$ is finitely satisfiable in M.
A theory T *has the f.s.-dichotomy* if given $A \mathop{\smile\!\!\!\!\!|}^{fs}_{M} B$, then for any c, $cA \mathop{\smile\!\!\!\!\!|}^{fs}_{M} B$ or $A \mathop{\smile\!\!\!\!\!|}^{fs}_{M} Bc$.

Theorem ([4])

If T does not have the f.s.-dichotomy, then T admits a pre-coding configuration, *and so is not monadically NIP.*
If T has the f.s.-dichotomy, then models of T admit a nice linear decomposition into substructures.

THE F.S.-DICHOTOMY

- Recall: given $A \mathop{\smile\!\!\!\!\!\mid}^{fs}_M B$, then for any c, $cA \mathop{\smile\!\!\!\!\!\mid}^{fs}_M B$ or $A \mathop{\smile\!\!\!\!\!\mid}^{fs}_M Bc$.
- Implies dependence is trivial* and transitive on singletons.
- *: If $A \not\mathop{\smile\!\!\!\!\!\mid}^{fs}_M B$, then $A \not\mathop{\smile\!\!\!\!\!\mid}^{fs}_M b$ for some $b \in B$.
  If $C \supset M$ is *large* (i.e. realizes all types over M), and $A \not\mathop{\smile\!\!\!\!\!\mid}^{fs}_{M \subset C} B$, then $a \not\mathop{\smile\!\!\!\!\!\mid}^{fs}_{M \subset C} B$ for some $a \in A$.
- So if we work over a large $C \supset M$, dependence gives a quasi-order.
- Why do we need C? Stationarity: If $p \in S(C)$ is fin. sat. in M, then for any $D \supset C$ there is a unique extension p over D that is fin. sat. in M. (No assumption of f.s.-dichotomy.)

M-F.S. SEQUENCES

Definition

Given a model M, $(a_i : i \in I)$ is an *M-f.s. sequence* if $a_i \mathop{\smile\!\!\!\!\!\!\,|}^{fs}_{M} \{a_{<i}\}$.

- Similar to Morley sequences. If also indiscernible, then a special case of Morley sequences.

Theorem (No assumption of f.s.-dichotomy)

Given an indiscernible sequence $\mathcal{I} \subset \mathfrak{C}$, we can find some model M so that $\mathcal{I}$ is an M-f.s. sequence.
Furthermore, we can find large $C \supset M$ so that $\mathcal{I}$ remains indiscernible and M-f.s. over C.

- Finite satisfiability and M-f.s. sequences seem like useful notions in arbitrary theories.

NON-STRUCTURE

- If the f.s.-dichotomy fails, we want a failure of monadic NIP.
- Given $\bar{a}, \bar{b}, c, M$ failing the f.s.-dichotomy, extend $\bar{a}\bar{b}$ to an M-f.s. sequence over a large $C \supset M$.
- By automorphisms, for $i < j$ find c_{ij} so $\text{tp}(\bar{a}\bar{b}c) = \text{tp}(\bar{a}_i\bar{b}_jc_{ij})$ (so c_{Ij} is non-generic over $\bar{a}_i\bar{b}_j$) but is reasonably generic over the rest of the sequence.
- This gives a *pre-coding configuration* as below.

Definition

A *pre-coding configuration* is an indiscernible sequence $(\bar{d}_i : i \in I)$ and formula $\phi(\bar{x}, \bar{y}, z)$ such that for every $s < t$, there is c_{st} satisfying the following.

1. $\models \phi(\bar{d}_s, \bar{d}_t, c_{st})$
2. $\not\models \phi(\bar{d}_u, \bar{d}_t, c_{st})$ for $u < s$
3. $\not\models \phi(\bar{d}_s, \bar{d}_v, c_{st})$ for $t < v$

NON-STRUCTURE CONTD.

- After Ramsey's theorem, combinatorial arguments give coding in a unary expansion.
- The unary expansion is used to "recover the rows" $\bar{d}_i$ from the first element, so the tuples can be replaced by singletons.
- Shelah's unary expansion is non-explicit.

Linear decompositions

Definition

A *linear decomposition* of M is a partition $M = \sqcup_i A_i$ and a model N (not necessarily in M) such that $(A_i : i \in I)$ is an N-f.s. sequence.

- From the f.s.-dichotomy, we can extend partial linear decompositions one point at a time.
- Example: DLO

- Somewhat like one step of the tree decomposition, although the parts are ordered.
- Linear decompositions give an *order-congruence* over any large $C \supset N$.

Main theorem

Theorem (B, Laskowski)

The following are equivalent.

1. *T is monadically NIP.*
2. *T does not admit coding in a unary expansion.*
3. *T does not admit a pre-coding configuration.*
4. *T has the f.s.-dichotomy.*
5. *Partial linear decompositions of models of T extend to full linear decompositions.*
6. *T is dp-minimal and indiscernible trivial.*

- From Shelah's results, we still need (5) $\Rightarrow$ (1), and to show the equivalence with (6).
- We also redo the non-structure part of Shelah's proof more carefully to get our result about finite structures.

FROM DECOMPOSITIONS TO MONADIC NIP

- Given an indiscernible sequence $\mathcal{I} = (a_i : i \in I)$, we consider a partition of $\mathfrak{C}$ with each a_i in a different part.
- We choose a finite subset of that partition, and count the number of types realized over it.
- If T has IP, then by taking $\mathcal{I}$ sufficiently long and shattered, we must realize unboundedly many types.
- If T can extend $\mathcal{I}$ to a linear decomposition over M, then doing so will realize few types ($\beth_2(\aleph_0)$).
- This uses that each part is finitely satisfiable in M, so few types in each part, and the parts form an order-congruence.
- So few *quantifier-free* types realized in any monadic expansion of T.
- But we can bound the number of types realized in terms of the number of q.f.-types realized (by applying $\beth_{\omega+1}$).
- This type-counting seems similar to linear clique width (cf [2]).

INDISCERNIBLES

Definition

T is *dp-minimal* if whenever $\mathcal{I}$ is dense indiscernible, then $\mathcal{I}$ splits into at most three parts indiscernible over a parameter c, with one part initial, one a singleton, and one terminal.
T is *indiscernible trivial* if whenever $\mathcal{I}$ is indiscernible over each $a \in A$, then $\mathcal{I}$ is indiscernible over A.

- Thanks to Pierre Simon for suggesting this characterization.
- Example: DLO

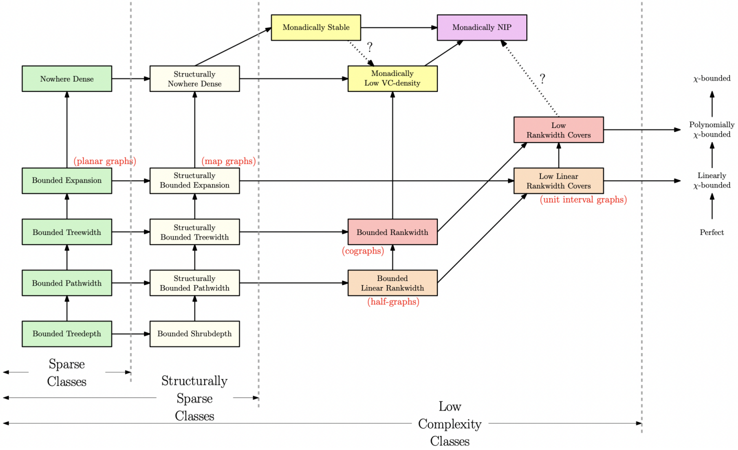
- Fairly easy that if have these properties, then can't have a pre-coding configuration.
- If T is monadically NIP, linear decompositions show its models "look like DLO".

A dividing line

- Monadic NIP should be a dividing line for several properties of hereditary classes.
- Should provide a general setting for decompositions as in structural graph theory.
- For example, see recent work on twin-width and ordered graph classes, where it coincides with monadic NIP [5].
- Also see work on sparse graph classes, started by Nešetril and Ossona de Mendez.

A PICTURE OF GRAPH CLASSES [3]

Classes of graphs with low complexity



- "Structurally P" closes P under definability in unary expansions.

Homogeneous structures

Definition

Given a structure M, the *growth rate* of M is a function $\varphi_M(n)$ counting the (unlabeled) isomorphism types of n-substructures.

- We add monadic NIP to a question of Macpherson.

Conjecture

Let M be a homogeneous ω-categorical structure. The following are equivalent.

1. *M is monadically NIP.*
2. *The growth rate of M is at most exponential.*
3. *$Age(M)$ is well-quasi-ordered by embeddability, i.e. there is no infinite antichain.*

- We prove non-structure results: $(2) \Rightarrow (1)$ and a weak form of $(3) \Rightarrow (1)$, just assuming QE.

The theorem

Theorem (B, Laskowski)

Suppose M has QE and is not monadically NIP. Then

1. *the growth rate of M is at least $(n/k)!$ for some $k \in \mathbb{N}$*
2. *there is some expansion M^* of M by ℓ unary predicates with $Age(M^*)$ not well-quasi-ordered*

- No uniform bounds on k, ℓ.

Coding finite graphs

- We want to encode bipartite graphs with n edges and n vertices in each part in $O(n)$-substructures of a unary expansion of M.
- By our characterization, if M is not monadically NIP, it admits a pre-coding configuration.
- Shelah showed how to then code bipartite graphs in an unspecified unary expansion.
- You only need to name the "columns" of the pre-coding configuration, which lets you recover the "rows" from any element [2].
- If $\psi(x, y, z)$ witnesses coding, we want to ensure ψ behaves the same in our finite structures as in $\mathfrak{C}$.
- We keep track of which elements are needed to witness (the failure of) quantifiers in ψ so we can include them in our finite structures.

Questions

Question

Can we give uniform bounds on k and ℓ in the last theorem? In particular, can we get rid of ℓ?
Can the linear decomposition be refined to a tree decomposition? [1]

Question

Can the quantifier-elimination for mutually algebraic theories be generalized to monadic stability?

Question

Is there a tree-decomposition for monadically stable structures more suited to finite combinatorics?
Does monadic stability imply low VC-density, i.e. $vc(\phi(\bar{x};\bar{y})) = |\bar{x}|$?
Is a hereditary graph class monadically stable iff it is definable in a unary expansion of a nowhere-dense class? [3]

REFERENCES I

[1] John T Baldwin and Saharon Shelah, *Second-order quantifiers and the complexity of theories.*, Notre Dame Journal of Formal Logic **26** (1985), no. 3, 229–303.

[2] Achim Blumensath, *Simple monadic theories*, Habilitation, 2008.

[3] Jaroslav Nešetřil, Patrice Ossona de Mendez, Roman Rabinovich, and Sebastian Siebertz, *Classes of graphs with low complexity: the case of classes with bounded linear rankwidth*, European Journal of Combinatorics **91** (2021), 103223.

[4] Saharon Shelah, *Monadic logic: Hanf numbers*, Around classification theory of models, 1986, pp. 203–223.

[5] Pierre Simon and Szymon Toruńczyk, *Ordered graphs of bounded twin-width*, arXiv preprint arXiv:2102.06881 (2021).